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Methods of Solution of the Velocity–Vorticity Formulation of the Navier–Stokes Equations

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Some methods are proposed for solving the Navier–Stokes equations for two-dimensional, incompressible, flow using the velocity-vorticity formulation. The main feature of the work is the solution of the equation of continuity using boundary-value techniques. This is possible because both of the velocity components are known at each boundary point. Some illustrative results are computed including some for heat convection inside a square cavity when one side is held at a constant temperature. © 1995 Academic Press, Inc.

1. INTRODUCTION

This paper is devoted to a study of some methods of solving the Navier–Stokes equations in the velocity–vorticity formulation. In this formulation the equations are expressed entirely in terms of the components of velocity and vorticity. In the general three-dimensional case this involves six equations, but we shall consider in detail only the case of two-dimensional flow in which there are only two velocity components and one component of vorticity. However, the extension to three dimensions can be made, since our main object is to consider methods by which the velocity field is derived from the vorticity field, at the same time ensuring that the equation of continuity is satisfied.

1.1. *Basic Equations*

The velocity–vorticity method has recently been reviewed by Gatski [1] who has given many references to work on this formulation. We may start with the basic equations for incompressible flow

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\rho^{-1}\nabla p + \nu \nabla^2 \boldsymbol{v} \qquad (1)$$

$$\nabla \cdot \boldsymbol{v} = 0, \qquad (2)$$

where $\boldsymbol{v}$ is the velocity vector, ρ is the density, p the pressure, and ν is the coefficient of kinematic viscosity. If we note the definition

$$\boldsymbol{\omega} = \text{curl } \boldsymbol{v} \qquad (3)$$

which defines the vorticity vector $\boldsymbol{\omega}$ then, taking the curl of (1) we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} = \nu \nabla^2 \boldsymbol{\omega}, \qquad (4)$$

which gives the equation of transport of vorticity. Equation (4) is central to all methods in the velocity–vorticity formulation and the methods of approach vary only in the manner in which the velocity components are determined from $\boldsymbol{\omega}$. We shall consider here only the steady-state analogue of (4) in which $\partial \boldsymbol{\omega}/\partial t \equiv \boldsymbol{0}$. This gives a boundary-value problem for the determination of the components of $\boldsymbol{\omega}$ from given approximations to the velocity vector $\boldsymbol{v}$. In the present approach this determination is carried out by well-known finite-difference methods. However, the methods of determining the velocity components differ considerably from previous approaches. In the present work we shall present methods having some novel features.

1.2. *Previous Investigations*

Gatski has noted three separate types of methods for determining the velocity components. In the first of these they are obtained by making use of a fundamental solution procedure, namely the Biot–Savart law. This has largely been developed by Wu and co-workers [2], although the first application of this method to the two-dimensional flow past a circular cylinder was given by Payne [3]. The second method is that in which the velocity components are determined directly from the defi-

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nitions (3) of the vorticity. Gatski, Grosch, and Rose [4, 5] have described these methods in two and three dimensions and there are numerous other papers noted by Gatski [1]. The third method is that in which Eqs. (2) and (3) are used to derive second-order Poisson-type differential equations for each of the separate velocity components. These may then be solved by boundary-value techniques, since boundary conditions are given for all the velocity components. The first work on this type of method was performed by Fasel [6] and by Cook [7].

1.3. *Present Work*

In the present paper we propose a method which is somewhat similar in principle to that of Refs. [4, 5] but which involves solution principles which are basically new. In the two-dimensional case which we consider there is only one equation for the scalar vorticity which, together with the equation of continuity, serves to define the velocity components once an approximation to the vorticity is obtained. These two equations are combined to obtain one second-order equation for one of the velocity components by differentiation, following the methods of Refs. [6, 7], but the other component is determined from the equation of continuity itself by solving it as a first-order equation, but using boundary-value techniques. It is, in fact, this solution procedure which provides the new features of our method. Moreover, by ensuring that the equation of continuity is solved in an undifferentiated form we clearly ensure that it is satisfied, at least to the order of the difference approximations involved. We shall first illustrate the method by a very simple one-dimensional example and then apply it to a trial two-dimensional problem involving the linear biharmonic equation. Finally, the problem of heat convection in a square cavity in which one side is kept at a constant temperature is considered.

2. BASIC METHOD IN TWO DIMENSIONS

In the case of two-dimensional flow Eq. (4) simplifies to a single scalar equation for the component ζ of vorticity, where $\boldsymbol{\omega} = (0, 0, \zeta)$. We shall suppose that $\boldsymbol{\omega}$ has been made dimensionless with respect to a representative length d and a representative velocity U. The coefficient ν can be removed from (4) and the Reynolds number $R = Ud/\nu$ introduced into the equations. The velocity vector is made dimensionless by dividing by U and thus we define $\boldsymbol{\nu}/U = (u, v)$, where (u, v) are the dimensionless velocity components.

2.1. *Two-Dimensional Equations*

In two dimensions the dimensionless analogues of Eqs. (2) and (4) are, in the case of steady flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

and

$$\nabla^2 \zeta = R\left(u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right), \tag{6}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Eq. (3) reduces to the single scalar equation

$$\zeta = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{7}$$

Our restriction of the work to steady-state flow is of no special significance since our main purpose is to deal with the solution of (5) or (7), which hold equally for steady or unsteady flow.

Each of Eqs. (5) and (7) can be considered, in effect, as a one-dimensional first-order equation in one or the other of the variables u and v but with the conditions that u and v are given on all boundaries. There is thus more information available than is normally present for first-order equations and it raises the question as to whether such problems can be solved by boundary-value techniques. We have devised an appropriate technique which will be illustrated first of all by a simple one-dimensional example.

2.2. *One-Dimensional Illustration*

Consider the one-dimensional equation

$$u'(x) = r(x), \tag{8}$$

where the prime denotes differentiation with respect to x. We suppose that $r(x)$ is specified and we are required to find $u(x)$. The new point about the Navier–Stokes type of problem is that $u(x)$ is known at both ends of the variable domain in x rather than only one. If therefore we expand $u(x)$ as a Taylor series about a point $x = x_i$ of a uniform grid of length h, we find that

$$-3u_{i-1} + 4u_i - u_{i+1} = f_i, \tag{9}$$

where

$$f_i = 2hr_i - 2h^2r_i' + \tfrac{4}{3}h^3r_i'' + O(h^4). \tag{10}$$

This is a 3-point difference formula suitable for determining an approximation to $u(x)$ by boundary-value techniques at a set of grid points in the solution domain. The use of boundary-value methods is feasible because of the two-point boundary conditions and we suppose that sufficient terms on the right-hand side of (9) can be calculated to determine an approximation of sufficiently high order in h.

2.3. *Numerical Example*

We need only a very simple case of (8) to show that the method works. For a given $r(x)$ any suitable matrix inversion of (9) is satisfactory, since (9) defines a tridiagonal matrix which

TABLE I

Solutions of Eq. (8) with $r(x) = \sin x$, $u(0) = 0$, $u(\pi) = 2$

Approximation	$x = 0.25\pi$	$x = 0.5\pi$	$x = 0.75\pi$
A	0.3960	1.1341	1.7819
B	0.2943	1.0044	1.7142
C	0.2927	0.9997	1.7069
E	0.2929	1.0000	1.7071

is diagonally dominant. We have chosen the SOR iterative procedure in the form

$$u_i^{k+1} = (1-\omega)u_i^k + \frac{\omega}{4}(3u_{i-1}^{k+1} + u_{i+1}^k + f_i), \tag{11}$$

assuming that the components u_i ($i = 1, 2, \ldots, n$) are determined in the order of ascending i. The parameter ω is the relaxation factor and we know that the procedure (11) will be convergent for all ω such that $0 < \omega < 2$. We have taken the case $r(x) = \sin x$ with boundary conditions $u(0) = u(\pi) = 2$. Thus the solution of (8) is

$$u(x) = 1 - \cos x. \tag{12}$$

2.4. *Calculated Results*

Two sets of calculations have been carried out in the case of (9) and (10) with $r(x) = \sin x$ over the region $x = 0$ to $x = \pi$, the first with $h = \pi/20$ and the second with $h = \pi/40$. In both cases three approximations were obtained, corresponding to retaining one, two, and three terms respectively on the right-hand side of (10). For the first of these cases an approximation was obtained corresponding to various values of ω in the range $1 \le \omega \le 1.9$ in (11), in each case starting the iterative process from the initial assumption $u_i = 0$ ($i = 1, 2, \ldots, n$). The number of iterations to convergence, described by the test

$$\sum_{i=1}^{n} |u_i^{k+1} - u_i^k| < 0.0001 \tag{13}$$

was recorded. For $\omega = 1$ this number was 41, decreasing to 13 at $\omega = 1.4$ and then increasing, e.g., to 156 at $\omega = 1.9$. Some comparative results for the three solutions, denoted by A, B, and C, are given in Table I for the grid size $h = \pi/20$. The exact solution (12) is denoted by E and the results indicate the improvement of the approximations as more terms are taken on the right-hand side of (9). The approximations using the grid size $h = \pi/40$ are significantly better, the best of them being correct to almost five decimals.

3. THE NAVIER–STOKES PROBLEM

We now turn our attention to the solution of Eqs. (5)–(7). The vorticity transport equation (6) is approximated in the present work by means of standard second-order accurate central-difference formulae which are well known and do not need to be described. The main purpose is to consider the determination of the velocity components from Eqs. (5) and (7) and, in particular, the solution of (5) using the method of the previous section.

3.1. *Approximation to Eq. (5)*

Equation (5) can be approximated simply by identifying the term $\partial u/\partial x$ with the left-hand side of (8) and the term $-\partial v/\partial y$ with the right-hand side, both terms now being understood to represent functions of the variables x and y. Thus Eq. (9) must now be written in double-subscript form, namely

$$-3u_{i-1,j} + 4u_{i,j} - u_{i+1,j} = f_{i,j} \tag{14}$$

and $f_{i,j}$ must likewise be defined in (10) in terms of values at a given grid point i,j of a function $r(x, y) = -\partial v/\partial y$ and its derivatives with respect to x. In the present work we have retained all the stated derivatives in (10), since they can all be expressed in terms of second-order accurate central differences over a nine-point compact molecule. An appropriate approximation is found to be

$$\begin{aligned} f_{i,j} &= \tfrac{1}{3}(v_{i+1,j+1} - v_{i+1,j-1}) \\ &\quad - \tfrac{2}{3}(v_{i,j+1} - v_{i,j-1} - v_{i-1,j+1} + v_{i-1,j-1}). \end{aligned} \tag{15}$$

Because the function $r(x, y)$ in this problem has been obtained by second-order numerical differentiation, the error in (15) is $O(h^3)$. To preserve the $O(h^4)$ accuracy of (10) it is necessary to calculate $\partial v/\partial y$ to at least $O(h^3)$ and this destroys the compact nature of the molecule.

The set of Eqs. (14) is solved by the SOR procedure in a similar manner to the set (9), proceeding along rows for $i = 1, 2, \ldots, N$. The method is therefore equivalent to the method of lines. It is assumed during the course of these iterations that the components of the vector $f_{i,j}$ remain fixed at the start of the iterations, i.e., $f_{i,j}$ is not updated during the iterations. In fact it was found to be efficient to carry out one complete iteration only along each of the lines in the x direction throughout the whole field and then to proceed to the solution for the component v.

3.2. *Solution Procedure for $v(x, y)$*

A similar procedure cannot be used for determining $v(x, y)$ directly from (7). This would involve applying the method of lines in the x direction also to determine v and so far we have not succeeded in finding a stable process in the iterative determination of u and v. Thus to determine v we make use of the second-order equation obtained by eliminating u from Eqs. (5) and (7), namely

$$\nabla^2 \nu - \frac{\partial \zeta}{\partial x} = 0. \quad (16)$$

This is now approximated by the customary second-order accurate central-difference formulae. This completes one formulation of the problem.

3.3. *Alternative Method of Solution*

We can clearly also solve Eq. (5) for $\nu(x, y)$ by writing it in the form $\partial \nu/\partial y = -\partial u/\partial x$, which can then be approximated in the form

$$-3\nu_{i,j-1} + 4\nu_{i,j} - \nu_{i,j+1} = g_{i,j}, \quad (17)$$

where

$$\begin{aligned} g_{i,j} &= \tfrac{1}{3}(u_{i+1,j+1} - u_{i-1,j+1}) \\ &\quad + \tfrac{2}{3}(u_{i-1,j} - i_{i+1,j} - u_{i+1,j-1} + u_{i-1,j-1}). \end{aligned} \quad (18)$$

The set of Eqs. (17) are then solved successively by iterative methods along lines of constant i, each solution covering the values $j = 1, 2, ..., N$. Since two-point boundary values of $\nu(x, y)$ are given in the direction of y for each value of x, the solution of (17) can be carried out by any method of matrix inversion. In the present work, as mentioned, an iterative procedure was used.

Along with the solution procedure of determining a solution of (17) for $\nu_{i,j}$ we use a second-order equation for determining an approximation to $u(x, y)$. This is easily found from (5) and (7) to be

$$\nabla^2 u + \frac{\partial \zeta}{\partial y} = 0 \quad (19)$$

and this is solved by approximating all derivatives using central differences. The finite-difference equations obtained in this way are solved subject to Dirichlet conditions, since boundary values for $u(x, y)$ are known.

4. NUMERICAL EXAMPLES

The numerical examples used to illustrate these methods in the two-dimensional case have both been solved over the domain of a unit square using a grid of size h. In general the solution procedure adopted was to perform one iteration of the vorticity equation for $i, j = 1, 2, ..., N$, one iteration of the second order equation for $\nu(x, y)$ again for $i, j, = 1, 2, ..., N$ and then one iteration of Eq. (14) for $u(x, y)$ with $i = 1, 2, ..., N$ along each line of constant j. This procedure was repeated until overall convergence, defined by tests similar to (13), was obtained. The alternative formulation described in Section 3.3 was implemented in a similar manner.

TABLE II

Computed Values of $\zeta(0.5, 0.5)$ for Four Model Solutions of Eq. (20)

h	A	B	C	D
1/20	3.626	3.626	3.632	3.492
1/40	3.549	3.549	3.554	3.515
1/60	3.535	3.535	3.539	3.520

4.1. *Solution of the Biharmonic Equation*

As a simple illustration of the two-dimensional method we consider the solution inside the unit square, $0 \le x \le 1$, $0 \le y \le 1$, of the equations

$$\nabla^2 \zeta + 100 = 0, \quad \nabla^2 \psi + \zeta = 0 \quad (20)$$

with

$$\begin{aligned} \psi &= \frac{\partial \psi}{\partial x} = 0 \quad \text{when } x = 0, 1, \\ \psi &= \frac{\partial \psi}{\partial y} = 0 \quad \text{when } y = 0, 1. \end{aligned} \quad (21)$$

In this problem the forcing term in the vorticity equation depends neither on the stream function nor the Reynolds number, but the function ζ is dependent on ψ by means of the calculation of its boundary values on the unit square in terms of values of ψ within the square. We have considered four model processes, denoted by A, B, C, and D, all of which use the first of (20) in finite-difference form to determine the vorticity.

In model A we have used the formulation of Sections 3.1 and 3.2 and in model B the alternative formulation of Section 3.3 to compute $u(x, y)$ and $v(x, y)$. In model C the function $u(x, y)$ is determined from (19) while $\nu(x, y)$ is determined from (16), using two-dimensional boundary-value methods for both. In all of the models A, B, and C the boundary vorticity is calculated from (7) using three-point forward or backward difference formulae to approximate the appropriate derivative normal to the boundary. Finally, in model D we utilize the usual vorticity–stream function formulation of (20), using the standard central finite-difference approximation to determine values of ψ within the square and with boundary values of ζ calculated from the formula of Woods [8], namely

$$\zeta_B = -3\psi_I/h^2 - \tfrac{1}{2}\zeta_I, \quad (22)$$

where the subscript I denotes the first internal grid point along the inward normal to the boundary point B. This formula is second-order accurate.

4.2. *Computational Results*

We have computed results using the three grid sizes $h = 1/20$, $1/40$, and $1/60$. As may be expected, all methods give comparable results in accuracy. In Table II we have given computed values of $\zeta(0.5, 0.5)$ obtained from all four models. The h^2-extrapolated values from the $h = 1/40$ and $h = 1/60$ solutions are 3.521 from models A and B, 3.527 from the model C, and 3.524 from model D. These estimates would seem to be accurate within a decimal or two in the third place.

4.3. *Free Convection in a Square Cavity*

As a problem involving the full Navier–Stokes equations we consider the problem of free convection in a square cavity which has been used previously by many authors as a test of numerical methods. In terms of the stream function $\psi(x, y)$, vorticity $\zeta(x, y)$, and the temperature $T(x, y)$ within the region, $0 \leq x \leq 1$, $0 \leq y \leq 1$, the governing equations can be expressed as

$$\nabla^2\psi + \zeta = 0 \tag{23}$$

$$\nabla^2\zeta = \mathrm{Pr}^{-1}\left(u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}\right) - \mathrm{Ra}\frac{\partial T}{\partial x} \tag{24}$$

$$\nabla^2 T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}. \tag{25}$$

Here, Pr is the Prandtl number, Ra is the Rayleigh number, and all variables are assumed to be dimensionless. If C denotes the unit square and $\mathbf{n}$ is the outward normal to it at any point, the boundary conditions are

$$\psi = \partial\psi/\partial n = 0 \quad \text{on } C; \tag{26a}$$

$$T = 1 \quad \text{when } x = 0, \quad T = 0 \quad \text{when } x = 1; \tag{26b}$$

$$\partial T/\partial y = 0 \quad \text{when } y = 0, 1. \tag{26c}$$

Bench-mark solutions for this problem have been given by de Vahl Davis [9] and various comparison solutions have been described and discussed by de Vahl Davis and Jones [10]. Thus we have the opportunity of making quite detailed comparisons with these results. Equations (23) and (24) may easily be compared with Eqs. (5)–(7). Thus, with the customary definitions $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, Eqs. (5) and (7) represent (23). Also, (24) may be identified with (6) if we add the term $-\mathrm{Ra}\,\partial T/\partial x$ to the right-hand side and put $R = \mathrm{Pr}^{-1}$. This may then be expressed in central differences in the form

$$\left(1 - \frac{1}{2}h\,u^*_{i,j}\right)\zeta_{i+1,j} + \left(1 - \frac{1}{2}h\,v^*_{i,j}\right)\zeta_{i,j+1} + \left(1 + \frac{1}{2}h\,u^*_{i,j}\right)\zeta_{i-1,j} + \left(1 + \frac{1}{2}h\,v^*_{i,j}\right)\zeta_{i,j-1} - 4\zeta_{i,j} + \frac{1}{h}\mathrm{Ra}(T_{i+1,j} - T_{i-1,j}) = 0, \tag{27}$$

where $u^* = \mathrm{Pr}^{-1}u$, $v^* = \mathrm{Pr}^{-1}v$. The central-difference approximation to (25) may be obtained from (27) by writing $\zeta = T$ and putting $\mathrm{Pr} = 1$, $\mathrm{Ra} = 0$.

With the changes noted in solving (24) rather than (6) and with the additional solution required for (25), we have obtained approximate solutions using the models A and B described in Section 4.1. The difference equations (27) have been used in all cases inside the unit square. The boundary conditions for ζ have been calculated as in Section 4.1. The boundary conditions for $T(x, y)$ on $y = 0, 1$ are obtained by approximating (26c) using central-difference formulae and using the equation obtained to eliminate the external value which is introduced when (27), with $\zeta = T$, $\mathrm{Pr} = 1$, $\mathrm{Ra} = 0$, is applied on either $y = 0, 1$.

4.4. *Computational Results*

The two models A and B have been solved for the case $\mathrm{Pr} = 0.71$ considered by de Vahl Davis [9] and for the Rayleigh numbers $\mathrm{Ra} = 10^3$ and 10^4 using the $O(h^4)$ form of (15). Three grid sizes $h = 1/20, 1/40$, and $1/60$ were used to obtain solutions. Results for the mid-point values $|\psi(\frac{1}{2}, \frac{1}{2})|$ and $\zeta(\frac{1}{2}, \frac{1}{2})$ from these solutions are shown for model A in Table III and for model B in Table IV. The bench-mark value given by de Vahl Davis for $|\psi(\frac{1}{2}, \frac{1}{2})|$ is 1.174 at $\mathrm{Ra} = 10^3$ and 5.071 at $\mathrm{Ra} = 10^4$. The values corresponding to these, obtained by h^2 extrapolation from the $h = 1/40$ and $h = 1/60$ solutions of Table III, are 1.174 and 5.072, respectively; the similar h^2-extrapolated values from Table IV are respectively 1.174 and 5.076. Thus, bearing in mind the high level of accuracy claimed for the bench-mark solutions, the present solution procedures seem capable

TABLE III

Values of $|\psi|$ and ζ at the Mid-point of the Cavity for Heat Convection Problems for $\mathrm{Pr} = 0.71$ Using Model A

	$\mathrm{Ra} = 10^3$		$\mathrm{Ra} = 10^4$	
h	$\lvert\psi(\frac{1}{2}, \frac{1}{2})\rvert$	$\zeta(\frac{1}{2}, \frac{1}{2})$	$\lvert\psi(\frac{1}{2}, \frac{1}{2})\rvert$	$\zeta(\frac{1}{2}, \frac{1}{2})$
$\frac{1}{20}$	1.260	33.07	6.595	121.23
$\frac{1}{40}$	1.196	32.28	5.461	103.32
$\frac{1}{60}$	1.184	32.13	5.245	99.85

TABLE IV

Values of $|\psi|$ and ζ at the Mid-point of the Cavity for Heat Convection Problems for $\mathrm{Pr} = 0.71$ Using Model B

	$\mathrm{Ra} = 10^3$		$\mathrm{Ra} = 10^4$	
h	$\lvert\psi(\frac{1}{2}, \frac{1}{2})\rvert$	$\zeta(\frac{1}{2}, \frac{1}{2})$	$\lvert\psi(\frac{1}{2}, \frac{1}{2})\rvert$	$\zeta(\frac{1}{2}, \frac{1}{2})$
$\frac{1}{20}$	1.252	32.96	5.641	103.57
$\frac{1}{40}$	1.194	32.25	5.225	98.73
$\frac{1}{60}$	1.183	32.12	5.142	97.82

TABLE V

Properties of the Solution of the Heat Convection Problem for Pr = 0.71, Ra = 10^3 for Model *A*

h	u_{max} $y(x = 0.5)$	v_{max} $x(y = 0.5)$	Nu_0	Nu_{max} $y(x = 0)$	Nu_{min} $y(x = 0)$
$\frac{1}{20}$	3.596	3.654	1.114	1.488	0.700
	0.816	0.178		0.090	1
$\frac{1}{40}$	3.631	3.679	1.116	1.500	0.694
	0.813	0.178		0.088	1
$\frac{1}{60}$	3.640	3.688	1.117	1.503	0.692
	0.813	0.178		0.088	1
Bench-mark	3.649	3.697	1.117	1.505	0.692
Solution	0.813	0.178		0.092	1

TABLE VII

Properties of the Solution of the Heat Convection Problem for Pr = 0.71, Ra = 10^3 for Model *B*

h	u_{max} $y(x = 0.5)$	v_{max} $x(y = 0.5)$	Nu_0	Nu_{max} $y(x = 0)$	Nu_{min} $y(x = 0)$
$\frac{1}{20}$	3.610	3.614	1.116	1.500	0.696
	0.813	0.179		0.095	1
$\frac{1}{40}$	3.633	3.670	1.116	1.502	0.693
	0.813	0.179		0.089	1
$\frac{1}{60}$	3.641	3.683	1.117	1.504	0.692
	0.813	0.179		0.083	1
Bench-mark	3.649	3.697	1.117	1.505	0.692
Solution	0.813	0.178		0.092	1

of giving good accuracy. We may note that the determination of the stream function is not an integral part of either of the model *A* or *B* procedures. The values given in Tables III and IV were obtained afterwards from each of the corresponding solutions for ζ by solving (23) subject to the conditions of (26a).

Some further representative properties of the model *A* solutions are given in Tables V and VI and the corresponding properties of the model *B* solutions are presented in Tables VII and VIII. The values of the Nusselt number Nu are all appropriate to the end $x = 0$ of the cavity and are calculated from the formula

$$\mathrm{Nu}(y) = -(\partial T/\partial x)_{x=0}. \qquad (28)$$

The value Nu_0 is the integrated mean value of $\mathrm{Nu}(y)$ over the range $y = 0$ to $y = 1$. The tendencies of all the properties in these tables as the grid size is reduced are generally consistent with the bench-mark values of de Vahl Davis, which are also given. The results also are in reasonable agreement with results for the solution of the same problem given by Dennis and Hudson [11], bearing in mind the completely different solution procedures adopted in the two investigations. In assessing the comparisons it should, however, be borne in mind that a number of the properties given in these tables have been obtained by interpolation of the corresponding solutions.

5. SUMMARY AND CONCLUSIONS

We have presented two methods, methods *A* and *B*, in which the equation of continuity for the two-dimensional motion of incompressible fluids is solved by iterative techniques based on boundary-value methods. This is possible because conditions are given for the velocity components at all points on the boundary of a closed domain. Note that the two methods do not always give identical results. This is because of the truncation errors inherent in the finite-difference approximations. However, one would expect that the difference between the results should decrease as h is reduced. Examination of Tables III–VI indicates that this is indeed the case. It may be concluded by comparison of the results obtained with other available solutions that the methods give satisfactory results.

In the two models considered, one velocity component is obtained by solving the first-order equation of continuity and the other is obtained from a second-order equation. This is

TABLE VI

Properties of the Solution of the Heat Convection Problem for Pr = 0.71, Ra = 10^4 for Model *A*

h	u_{max} $y(x = 0.5)$	v_{max} $x(y = 0.5)$	Nu_0	Nu_{max} $y(x = 0)$	Nu_{min} $y(x = 0)$
$\frac{1}{20}$	16.59	20.07	2.206	3.567	0.599
	0.827	0.117		0.165	1
$\frac{1}{40}$	16.34	19.80	2.239	3.488	0.590
	0.824	0.118		0.151	1
$\frac{1}{60}$	16.25	19.70	2.242	3.511	0.587
	0.823	0.119		0.148	1
Bench-mark	16.18	19.62	2.238	3.528	0.586
Solution	0.823	0.119		0.143	1

TABLE VIII

Properties of the Solution of the Heat Convection Problem for Pr = 0.71, Ra = 10^4 for Model *B*

h	u_{max} $y(x = 0.5)$	v_{max} $x(y = 0.5)$	Nu_0	Nu_{max} $y(x = 0)$	Nu_{min} $y(x = 0)$
$\frac{1}{20}$	16.41	19.67	2.329	3.746	0.589
	0.825	0.119		0.141	1
$\frac{1}{40}$	16.28	19.59	2.253	3.547	0.583
	0.824	0.122		0.144	1
$\frac{1}{60}$	16.23	19.60	2.246	3.531	0.584
	0.824	0.120		0.145	1
Bench-mark	16.18	19.62	2.238	3.528	0.586
Solution	0.823	0.119		0.143	1

TABLE IX

Average Errors per Solution Point in the Equation of Continuity and the Definition of Vorticity for the Heat Convection Problem for Pr = 0.71 Using Model *B*

		Average absolute error		Average scaled error	
Ra	h	S_1	S_1^*	S_2	S_2^*
10^3	$\frac{1}{20}$	0.104	1.258	0.016	0.068
	$\frac{1}{40}$	0.032	0.344	0.009	0.019
	$\frac{1}{60}$	0.015	0.163	0.006	0.009
10^4	$\frac{1}{20}$	1.011	10.88	0.044	0.112
	$\frac{1}{40}$	0.347	2.859	0.023	0.034
	$\frac{1}{60}$	0.175	1.302	0.015	0.016

derived by using the equation of continuity to eliminate one velocity component from a differentiated form of the first-order equation defining the scalar vorticity ζ. It would presumably be possible to employ the methods of the present paper to obtain one velocity component directly from the first-order equation which defines ζ and the other component from one or other of Eqs. (16) or (19) as appropriate. We have not investigated this question here.

It does, however, seem worthwhile to give some verification, in one of the examples considered in the present paper, that the computed velocity components and vorticity do actually satisfy Eqs. (5) and (7) to an acceptable level of accuracy. Some tests have therefore been carried out on the numerical solutions obtained in the case of the heat convection problem by calculating the quantities

$$E = |\partial u/\partial x + \partial v/\partial y|, \quad E^* = |\zeta - \partial v/\partial x + \partial u/\partial y|, \tag{29}$$

at each point of the solution domain. In order to test the whole domain, the quantities in (29) are summed over all grid points and the sums are then divided by the total number of internal grid points, thus giving an average error per point for each quantity.

Some typical results are given in Table IX for model *B*. Here the quantities S_1 and S_1^* are defined by

$$S_1 = \frac{1}{N}\sum_N E, \quad S_1^* = \frac{1}{N}\sum_N E^*, \tag{30}$$

where E and E^* are defined in (29) and the summations extend over all internal grid points N. These sums give the average absolute error per grid point, but it is perhaps more realistic to relate this error in some way to the absolute values of the quantities involved in defining it. Thus, we have also recorded scaled estimates of the average error, defined by

$$S_2 = \frac{1}{N}\sum_N \left\{E \Big/ \left(\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right|\right\}\right.,$$

$$S_2^* = \frac{1}{N}\sum_N \left\{E^* \Big/ \left(|\zeta| + \left|\frac{\partial v}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right|\right)\right\}.$$

These are, of course, considerably smaller than the average absolute error per point. However, the main point is that, however the error is estimated, it decreases with grid size. This is entirely consistent with the results displayed in Tables VII–VIII. The corresponding results for model *A* are similar and consistent with the results of Tables V–VI.

We have also computed values of S_1, S_1^*, S_2, and S_2^* from solutions obtained by model *C* (solving the two second-order equations (16) and (19) for u and v) and by model *D* (stream function-vorticity). As one would expect, those for model *D* were virtually zero. More interesting, however, is that those for model *C* were significantly higher than those given in Table IX, especially at the highest Rayleigh number. For example with h = 1/20 model *C* gives S_1 = 3.73, S_1^* = 16.44, S_2 = 0.17, and S_2^* = 0.15. With h = 1/40 the corresponding values are 1.02, 4.38, 0.059, and 0.046, respectively. These figures suggest that the equation of continuity is better satisfied by models *A* and *B* than by model *C*.

It is difficult to generalise on the question of efficiency. Cpu times were recorded for each model. Computing times were generally of the same order for any particular case and no particular model proved to be the most efficient overall.

We have presented an alternative approach to solving the equation of continuity which appears to give satisfactory results for the problems considered. One advantage of the method over the usual stream function formulation is that it should be possible to apply it to problems in three dimensions. This will be investigated in due course.

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